Basics

Index Notation 1 Whenever a quantity is summed over an index which appears exactly twice in each term in the sum, we leave out the summation sign. Such an index is called a *dummy index*. 2 An index appearing only once is called a *free index*. 3 No index may appear three times in a term. But one index can appear multiple times in an equation. A term is the basic unit in index notation. For example,

$$x_{ik}y_{jk}+a_{ik}b_{ik}=a_i+b_j \quad \text{means} \quad \sum_k x_{ik}y_{jk}+\sum_k a_{ik}b_{ik}=a_i+b_j$$

Definitions The *Kronecker delta* $\delta_{ij} := 1$ iff $i=j$ else 0.

We define the *Levi-Civita symbol* ε_{ijk} for $1 \le i,j,k \le 3$ to be 1 0 iff (i,j,k) is not a permutation of $(1,2,3)$. 2 1 iff $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$. 3 -1 iff $(i,j,k) \in \{(1,3,2),(2,1,3),(3,2,1)\}$. That is, ε_{ijk} equals to the parity of the permutation $(1,2,3) \to (i,j,k)$. Similarly, in 2D, ε_{ij} is 1 whenever $(i,j) = (1,2)$, -1 when $(i,j)=(2,1)$, and 0 otherwise.

1 The *double dot product* is $\mathbf{A} : \mathbf{B} := \sum_i \sum_j A_{ij}B_{ij}$ L1. 2 When one operand of the dot product is a matrix, it is interpreted as matrix multiplication. 3 If we apply gradiant to a vector-valued function $\mathbf{f}(\mathbf{x})$, then we are putting the gradiant of each component of it into a column of the result matrix: $\nabla\mathbf{f} := [(\nabla f_1)^T, \dots, (\nabla f_n)^T]$ L2.

1 The *divergence* of $\mathbf{P}$, $\nabla \cdot \mathbf{P} := \sum_i \frac{\partial P_i}{\partial x_i}$, as if $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$. It represents the source/sink of a $\mathbf{v}$ field. NOTE Distinguish $\nabla \cdot \mathbf{P}$ vs $\nabla\mathbf{P}$ R2! 2 The *Laplace operator (Laplacian)* is defined as the divergence of the gradiant of function f, $\nabla^2 f := \nabla \cdot \nabla f$. Rarely, we may use Δ for it. 3 The *curl* of $\mathbf{P}$ is defined as $\nabla \times \mathbf{P} = (\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, -(\frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3}), \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2})$. It represents the vorticity of a $\mathbf{v}$ field.

Facts 1 The area of a parallellogram equals $|\mathbf{a} \times \mathbf{b}|$. 2 The volume of a parallelpiped equals $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = \det(\mathbf{a},\mathbf{b},\mathbf{c})$. 3 $\mathbf{a} \times \mathbf{b} = \varepsilon_{ijk}\mathbf{e}_i a_j b_k$. 4 $\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$.

Change of basis

For vectors and matrices.

For vectors Let U be a finite n-dimensional vector space over field F. Let $(\mathbf{u}_i)_{i=1}^n$ and $(\mathbf{v}_i)_{i=1}^n$ be two ordered bases of U. Let $\mathbf{x}$ be any vector in U. Then, there exists two unique scalar sequences $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ such that $\sum_{i=1}^n a_i\mathbf{u}_i = \sum_{i=1}^n b_i\mathbf{v}_i$. They are also vectors in their own, in the vector space F^n. We want to find the change of basis function $T : (a_i)_{i=1}^n \mapsto (b_i)_{i=1}^n$, which exists since $(\mathbf{v}_i)_{i=1}^n$ is a basis.

T may be broken down into $T = f \circ g$ where $g((a_i)_{i=1}^n) := a_i\mathbf{u}_i$, and $f(\mathbf{x} \in U) := (b_i)_{i=1}^n$ such that $b_i\mathbf{v}_i = \mathbf{x}$. It is clear that both f and g are linear, and thus T is a linear function $F^n \to F^n$.

We then seek its unique matrix representation under the standard basis $(\mathbf{e}_i)_{i=1}^n$ of F^n: $[T] = [T(\mathbf{e}_1)^T, \cdots, T(\mathbf{e}_n)^T]$.

NOTE When $U = F^n$ and if we happen to use the standard basis of F^n, then $\mathbf{x}$ will appear exactly the same as $(a_i)_{i=1}^n$, which can be extremely confusing. We must constantly remind ourselves that T operates on the coordinate vectors and not directly on $\mathbf{x}$, which is the same vector regardless of the basis used to represent it as coordinates.

Change back Since $(\mathbf{u}_i)_{i=1}^n$ is a basis (it spans U and each $(a_i)_{i=1}^n$ is unique), T is invertible. Then, indeed, the function T^{-1} will be the change-back funcstion. As another way, one may swap the places of $(\mathbf{u}_i)$ and $(\mathbf{v}_i)$ in the above discussion.

For matrices Let f be a linear function $U \to U$. Under any basis $B = (\mathbf{u}_i)_{i=1}^n$, f has a unique matrix representation $[f]_B$. It is interesting to us how the matrix changes when we use another basis $B' = (\mathbf{v}_i)_{i=1}^n$ as the coordinate frame to get $[f]_{B'}$.

Let $\mathbf{x} \in U$ be any vector, and denote $f(\mathbf{x}) =: y$. Let $(a_i)_{i=1}^n$ be the coordinates of $\mathbf{x}$ under basis B, $(b_i)_{i=1}^n$ be the coordinates of $\mathbf{y}$ under B, and $(a'_i)_{i=1}^n, (b'_i)_{i=1}^n$ be the coordinates for them under B'. Now, consider the change-of-basis function T changing the coordinate frame under B to B', then, since f and $\mathbf{x}$ don't change for the basis, we should have

$$[f]_B T^{-1}(a'_i)_{i=1}^n = T^{-1}(b'_i)_{i=1}^n \quad \to \quad T[f]_B T^{-1}(a'_i)_{i=1}^n = (b'_i)_{i=1}^n$$

Thus $[f]_{B'} = T[f]_B T^{-1}$. NOTE If we are talking about orthognormal bases, then T will be an orthognormal matrix, resulting in $T^{-1} = T^T$.

Invariants Formally, we define an *invariant* to be any function on such matrices such that $f(M) = f(TMT^{-1})$ for all applicable matrices M, T. Three frequent invariants are 1 $\operatorname{tr} M$ 2 $\det M$ 3 $M_{11}M_{22} + M_{22}M_{33} + M_{11}M_{33} - M_{12}^2 - M_{23}^2 - M_{31}^2$.

Stress

Stress describes the forces present during the deformation of a material. It expresses the internal forces that neighbouring particles of a continuous material exert on each other.

Cauchy stress tensor Cauchy observed that the stress vector across a surface will always be a linear function of the surface's normal vector. The matrix for this function (under a fixed basis) can be treated as a tensor and called the Cauchy stress tensor.

For whatever reason, people chose to call the value of the Cauchy stress tensor function a *traction vector*, and use stress to refer to the matrix (tensor) of the function. NOTE In practice we assume the function takes normalized normals.

Under the standard basis, by the principle of conservation of angular momentum, the matrix can be shown to be symmetric L3. Thus, the traction vector can be computed by either $[\sigma]\mathbf{n}$ or $[\sigma]^T\mathbf{n}$. In the lecture the second way is used.

Normal & shear tractions For a traction vector, one can divide it into two components, 1 *Normal traction*, which is parallel to the surface normal $\mathbf{n}$, 2 *Shear traction*, which is perpendicular to the normal $\mathbf{n}$. One can easily calculate the normal traction $\mathbf{t}_n = (\mathbf{t} \cdot \mathbf{n})\mathbf{n}$. Then, the shear traction is $\mathbf{t}_s = \mathbf{t} - \mathbf{t}_n$. NOTE We may abuse notation sometimes to refer to the magnitude of the traction as traction. NOTE We also define the change in angle between two normalized vectors under a stress tensor σ to be (by symmetry), $2\mathbf{v}_1[\sigma]\mathbf{v}_2^T = 2\mathbf{v}_2[\sigma]\mathbf{v}_1^T$.

Infinitesimal strain The infinitesimal strain theory is an idealized theory in which one can arrive at simple formulae for deformation. Under this, the spatial and material coordinates are approximately the same, and we have the infinitesimal displacement/strain tensor $\epsilon = 1/2((\nabla_{\mathbf{x}}\mathbf{u})^T + \nabla_{\mathbf{x}}\mathbf{u})$ L4 and the infinitesimal rotation tensor $\omega = 1/2(\nabla_{\mathbf{x}}\mathbf{u} - (\nabla_{\mathbf{x}}\mathbf{u})^T)$, where $\mathbf{u}(\mathbf{x}, \dots)$ is the displacement vector field, and $\nabla_{\mathbf{x}}\mathbf{u} = [(\nabla_{\mathbf{x}}u_1)^T, \cdots, (\nabla_{\mathbf{x}}u_n)^T]$ R2.

About ϵ, an original location vector $\mathbf{x}$'s deformation will be described by it in such a way that $\mathbf{x}' = \epsilon\mathbf{x}$. We have 1 The diagonal elements of ϵ represent fractional length changes. E.g., if $\mathbf{x} \parallel \mathbf{e}_1$, then $\epsilon_{11} = (|\mathbf{x}'| - |\mathbf{x}|)/|\mathbf{x}|)$. 2 Thus, $\operatorname{tr}\epsilon = \nabla \cdot \mathbf{u}$ is the fractional change in volume. 3 Off-diagonal elements represent changes in angle. This is because, the angle, for unit vectors, is arccos of $\mathbf{x} \cdot \mathbf{x}' = \mathbf{x} \cdot (\epsilon\mathbf{x}) = \mathbf{x}\epsilon\mathbf{x}^T$. 4 As such, $2\epsilon_{ij}, i \neq j$ is the change in angle between $\mathbf{e}_i$ and $\mathbf{e}_j$ after the deformation. Also, given $\mathbf{p} \perp \mathbf{q}$, $2\mathbf{q}\epsilon\mathbf{p}^T$ is the change in angle between them.

Material vs spatial

Suppose we are interested in some physical property $\mathbf{P}$ of some material in space. 1 In material (Lagrangian) specification, the observer's eyes follows a particular particle, and the property is a function of the particle's initial location ξ and the time t: $\mathbf{P}(\xi, t)$. 2 In spatial (Eulerian) specification, the observer does not follow any particle but instead gives a global description of the space, resulting in function $\mathbf{P}(\mathbf{x}, t)$, giving the property for the particle at location $\mathbf{x}$ at time t.

Their link Suppose we are given a spatial description $\mathbf{P}(\mathbf{x}, t)$, and would like to use this to follow a specific particle to give a material description to it. Then, $\mathbf{x}$, for the particle, is a function: $\mathbf{x}(\xi, t)$, and $\mathbf{P}$ becomes $\mathbf{P}(\mathbf{x}(\xi, t), t)$. Specifically, if we want to find $\frac{\partial \mathbf{P}}{\partial t}$, then we need to use the chain rule to get

$$\frac{\partial \mathbf{P}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t}\frac{\partial t}{\partial t} = \frac{\partial \mathbf{P}}{\partial \mathbf{x}}\mathbf{v}(\xi, t) + \frac{\partial \mathbf{P}}{\partial t} = \sum_i \frac{\partial \mathbf{P}}{\partial x_i}v_i(\xi, t) + \frac{\partial \mathbf{P}}{\partial t}$$

Equations

Terms 1 *Steady state* means everything is constant w.r.t. time t. 2 *No flow* means velocity $\mathbf{v} = \mathbf{0}$. 3 *No strain, stress* means

the strain, stress tensors $\sigma, \epsilon = \mathbf{0}$.

of mass For the conservation of mass, 1 In spatial description, we have $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$. 2 In material description, it becomes $\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0$. 3 If the material is incompressible, i.e., ρ is constant, then the equation is reduced to $\nabla \cdot \mathbf{v} = 0$.

of momentum 1 The conservation of angular momentum results in σ's being symmetric R3. 2 As for that of linear momentum L5, we have $\rho\left(\frac{\partial^2 \mathbf{u}}{\partial t^2} = \frac{\partial \mathbf{v}}{\partial t}\right) = \mathbf{f} + \nabla \cdot \sigma = \sum_i \left(f_i + \sum_j \frac{\partial \sigma_{ji}}{\partial x_j}\right)$.

of energy The general form of conservation of energy in the lecture is $\frac{D\rho C_p T}{Dt} = \nabla \cdot k(\nabla T) + A + \sigma : \mathbf{D}$. , where : is R1 and the terms from the left to the right are 1 change in temperature with time 2 heat transfer by conduction (and radiation) 3 heat production (including latent heat) 4 heat generated by internal deformation.

Rheology We have rheology $\cdot$ deformation (ϵ) = stress (σ).

Elasticity 1 Elasticity means a material's deformation under a force will be restored when the force is removed. Under perfect elasticity, Hook's law states that the distance of deformation is proportional (linear) to the force applied: $\sigma = C : \epsilon$ R4. 2 Since it is linear, elasticity of a material is quantified by the *elastic modulus*, defined as $\delta := \frac{\text{stress}}{\text{strain}}$. 3 Young's modulus $E := \frac{\sigma_{11}}{\epsilon_{11}}$c, and Poisson's ratio $\nu := -\frac{\epsilon_{33}}{\epsilon_{11}}$. 4 In homogeneous and isotropic materials, Lamé's constants λ, μ define Hooke's law in 3D $\sigma = 2\mu\epsilon + \lambda \operatorname{tr}(\epsilon) I_{3\times 3}$. 5 With the Bulk (K) and shear (G) modulus: $-p = K\theta$ (isotropic) $\sigma'_{ij} = \frac{\sigma_{kk}}{3} = K\epsilon_{kk} = 2G\epsilon'_{ij}$ (deviatoric), where $\sigma_{ij} + p\delta_{ij} =: \sigma'_{ij}$. Thus, $K = \lambda + 2\mu/3$, where the second RHS term is compressional and the third is shear. 6 In Newtonian, general compressible fluids, $\sigma = (-p + \zeta\Delta)\mathbf{I} + 2\eta\mathbf{D}$, where D is the strain rate, $\Delta = D_{kk} = \nabla \cdot \mathbf{v}$. We have the Navier-Stokes equation $\nabla p + (\zeta + \eta)\nabla\Delta + \eta\nabla^2\mathbf{v} + \mathbf{f} = \rho\frac{D\mathbf{v}}{Dt} = \rho(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v})$ If the fluid is incompressible, then $\Delta = \nabla \cdot \mathbf{v} = 0$ and it's simplified to $\sigma = -pI + 2\eta D$, and we have the Navier-Stokes equation $-\nabla p + \eta\nabla^2\mathbf{v} + \mathbf{f} = \rho(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v})$.

Wave equation Substituting Lamé's constants formula into the equation of conservation of linear momentum R5, we have $\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathbf{f} + (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times \nabla \times \mathbf{u}$.

Interpolation

Let $(x_i, y_i)_{i=0}^N$ be $N+1$ data points. It can be shown that, provided $\forall i, j, x_i \neq x_j$, $\{(x_i^n)_{n=0}^N\}_{i=0}^N$ is linearly independent. Thus, the linear system $X\mathbf{a} = \mathbf{y}$, where X has these vectors as rows and $\mathbf{y} = (y_i)_{i=0}^N$, has a unique solution, $\mathbf{a} = (a_n)_{n=0}^N$ which is the coefficient vector of the unique polynomial of degree N passing through them.

Lagrange Let $\mathcal{P}_N$ be the set of all polynomials of degree N or less. It is a $(N+1)$-dimensional vector space. Lagrange found a basis $\{\ell_i(x)\}_{i=0}^N$, where $\ell_i(x) := \prod_{m\neq i} \frac{x - x_m}{x_i - x_m}$, and showed that $L(x) := \sum_{i=0}^N y_i \ell_i(x)$ is the unique interpolating polynomial.

Remainder Lagrange proved that, for any $f \in C^{N+1}[a, b]$, and datapoints $(x_i, y_i)_{i=0}^N$ that f passes through, the unique interpolation polynomial $P_N(x)$ results in a remainder, $R(x) := f(x) - P_N(x)$, satisfying $\forall x \in [a, b]$, $\exists c \in [a, b]$, $R(x) = \Psi(x) f^{N+1}(c)/(N+1)!$, where $\Psi(x) := \prod_{i=0}^N (x - x_i)$. NOTE Thus, $\Psi(x) \cdot \max(f^{N+1}(c))/(N+1)!$ is an upper bound of it.

Variants Observe that $\Psi(x)$ is much larger around a or b for evenly spaced datapoints. This, plus potentially large $f^{N+1}(c)$, gives very unstable results near the boundary. 1 Usually the choice of x_i is the only thing we can control. Lanczos found that $\max_{x_i \in [-1,1]} \Psi(x)$ attains the minimum when x_i are the roots of the Chebyshev polynomial $T_{N+1}(x)$, $x_i = \cos(\frac{2i-1}{2N}\pi)$. 2 We may also interpolate f piecewise to decrease the error, although it will make the interpolation function lose some smoothness.

Quadrature

1 Midpoint $I_M := \sum_{i=0}^{n-1} f\left(\frac{x_{i+1} + x_i}{2}\right)(x_{i+1} - x_i)$. 2 Trapezoidal $\sum_{i=0}^{n-1}\left(\frac{f(x_{i+1}) + f(x_i)}{2}\right)(x_{i+1} - x_i)$. 3 Simpson's $I_S := \frac{2}{3}I_M + \frac{1}{3}I_T = \sum_i \frac{(x_{i+1} - x_i)}{6}(f(x_i) + 4f(m) + f(x_{i+1}))$. 4 Weddle's $I_W = I_{S_2} + \frac{(I_{S_2} - I_S)}{15}$, where I_{S_2} is Simpson's applied on double amount of intervals. 5 Composite trapezoidal $\frac{\Delta x}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$. 6 Composite Simpson's $\frac{\Delta x}{3}\left[f(x_0) + 2\sum_{i=1}^{n/2-1} f(x_{2i}) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + f(x_n)\right]$. NOTE Composite Simpson actually uses 2 intervals as one, and the middle point is thus some x_i.

Error All these rules can be regarded as doing piecewise polynomial interpolation on f on evenly spaced datapoints, integrating f minus the polynomial, summing over the intervals, and using the Lagrange remainder to find the error. We may find 1 Trapezoidal: $-\frac{1}{12}\Delta x^2(b-a)\frac{1}{n}\sum_{i=0}^{n-1} f''(c_{x_i})$ 2 Midpoint: $\frac{1}{24}\Delta x^2(b-a)\frac{1}{n}\sum_{i=0}^{n-1} f''(c_{x_i})$. 3 Simpson's: since the error $I - I_T \approx -2(I - I_M)$, we imagine there's a better approximation I_S such that $I_S - I_T = -2(I - I_M)$, giving $I_S = \frac{2}{3}I_M + \frac{1}{3}I_T$. We have error $-\frac{\Delta x^4}{180}(b-a)f^{(4)}(c_x)$. 4 Weddle's: it would be a fuss to derive the exact one but it should be proportional to Δx^6.

ODE

To approximately solve (satisfying convergence as $\Delta t \to 0$ and correct qualitative behaviour) $y'(t) = f(t, y(t))$, we have 1 By the definition of derivative or the Taylor series, we have $y_{n+1} \approx y_n + \Delta t y'(t_n)$. This the the forward Euler. 2 By the definition of derivative, we may also say $y_{n+1} \approx y_n + \Delta t y'(t_{n+1})$, which is the backward Euler. 3 We may also say $y'(t_n) \approx \frac{y_{n+1} - y_{n-1}}{2\Delta t}$ and as such $y_{n+1} \approx y_{n-1} + 2\Delta t y'(t_n)$, which is leapfrog. 4 Recall the trapezoidal rule before, we may use it here to average the forward and backward Euler to obtain $y_{n+1} \approx y_n + \Delta t\frac{y'(t_n) + y'(t_{n+1})}{2}$.

Error 1 The *local error (LE)* is error committed at one step, assuming the previous step y_n is exact. Thus it is $y_{n+1} - y'_{n+1}$. 2 For example, using Taylor series, forward Euler has local error $(\frac{\Delta t^2}{2!}y''_n + \dots)$. 3 The *(local) truncation error (LTE)*, τ, is defined by the local error scaled down: $\tau := LE/(\Delta t)$. 4 A method is *consistent* if $\lim_{\Delta t \to 0} \tau = 0$. 5 The *global error* is $E := \max_{t_0 \leq t_n \leq T} \|y_n - y(t_n)\|$, only assuming the inital y_0 is exact. 6 A method *converges* iff $\lim_{\Delta t \to 0} E = 0$.

Stability Stability in numerical methods of solving ODEs have different definitions, but in general we would want the numerical methods to exhibit the same important properties as the exact solution. We have 1 A numerical method is said to be *A-stable*, if, when applied to the reference equation $y' = ky \wedge y(0) = 1$ for $k \in \mathbf{C}$, demonstrates that, provided $\Re(k) < 0$, $\lim_{t\to\infty}$ solution $\to 0$, the same property from the exact solution $y(t) = e^{kt}$.

2 A numerical method is *L-stable*, if it is A-stable, and that its stability function $\phi(x) \to 0$ as $z \to \infty$.

Adams-Bashforth For general ODE of form $y'(t) = f(y(t), t)$, according to the fundamental theorem of calculus, $y_{n+1} - y_n = \int_{t_n}^{t_{n+1}} f(y(t), t)\,\mathrm{d}t$.

The A-B schemes uses combinations of f_i for k many i's with $i \neq n+1$ to approximate the RHS integral to numerically solve the ODE. We have these AB schemes: 1 $k = 0$-step: $y_{n+1} = y_n + \Delta t f_n$. 2 $k = 1$-step: $y_{n+1} = y_n + \frac{\Delta t}{2}(f_n + f_{n+1})$. 3 $k = 2$-step: $y_{n+2} = y_{n+1} + \frac{\Delta t}{12}(-f_n + 8f_{n+1} + 5f_{n+2})$. 4 $k = 3$-step: $y_{n+3} = y_{n+2} + \frac{\Delta t}{24}(f_n - 5f_{n+1} + 19f_{n+2} + 9f_{n+3})$. 5 $k = 4$-step: $y_{n+4} = y_{n+3} + \frac{\Delta t}{720}(-19f_n + 106f_{n+1} - 264f_{n+2} + 646f_{n+3} + 251f_{n+4})$

Runge-Kutta Similar to the trapezoidal method, we have $y_{n+1} - y_n \approx \frac{1}{2}\Delta t(y'_n + y'_{n+1})$. However, here, $y'_{n+1} = f(y_{n+1}, t_{n+1})$, where y_{n+1} is not known.

In Runge-Kutta 2-step (RK2), we first use forward Euler to get $y^* \approx y_{n+1}$, then use this to calculate the RHS, and finally gives an approximation to y_{n+1} again. Perhaps surprisingly, RK2 is better in many ways than forward Euler. Specifically, it is L-stable.

But the most common one is RK4, which has $y_{n+1} = y_n + \frac{h}{6} \cdot (k_1 + 2k_2 + 2k_3 + k_4)$, where h is the step size, and 1 $k_1 = f(t_n, y_n)$ 2 $k_2 = f\left(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2}\right)$ 3 $k_3 = f\left(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2}\right)$ 4 $k_4 = f(t_n + h, y_n + hk_3)$.