STAT 210

Problem Set 4: Continuous Random Variables & Joint Distributions

Term: Fall 2026 |

Instructions

Complete all problems below. Show all step-by-step mathematical working. Write clearly and state key theorems used.

[EXR] Exercise 1 (Probability Density Functions)

Let X be a continuous random variable with probability density function (PDF) given by:

$$f_X(x) = \begin{cases} c(1-x^2) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the normalization constant c.
(b) Compute the cumulative distribution function (CDF), $F_X(x)$.
(c) Calculate the expected value $\mathbb{E}[X]$ and variance $\text{Var}(X)$.

Solution:

(a) Normalization Constant: By the fundamental property of PDFs, $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$:

$$c\int_0^1 (1-x^2)\,dx = c\left[x - \frac{x^3}{3}\right]_0^1 = c\left(1 - \frac{1}{3}\right) = \frac{2}{3}c = 1 \implies c = \frac{3}{2}$$

(b) Cumulative Distribution Function: For $x \in [0,1]$:

$$F_X(x) = \int_0^x \frac{3}{2}(1-t^2)\,dt = \frac{3}{2}\left(x - \frac{x^3}{3}\right) = \frac{3x - x^3}{2}$$

(c) Expectation and Variance:

$$\mathbb{E}[X] = \int_0^1 x \cdot \frac{3}{2}(1-x^2)\,dx = \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{3}{2}\left(\frac{1}{4}\right) = \frac{3}{8}$$

[EXR] Exercise 2 (Joint Probability Distributions)

Suppose X and Y have joint density $f(x,y) = 4xy$ for $0 \le x \le 1$ and $0 \le y \le 1$. Determine whether X and Y are independent random variables.

Solution:

We find marginal densities $f_X(x)$ and $f_Y(y)$:

$$f_X(x) = \int_0^1 4xy\,dy = 2x, \quad f_Y(y) = \int_0^1 4xy\,dx = 2y$$

Since $f(x,y) = 4xy = (2x)(2y) = f_X(x)f_Y(y)$, X and Y are **independent**.