Multivariable Calculus

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1 Vectors

I already know about vectors — I've been taught them in about five different courses so far. I'm skipping this.

2 Partial derivatives

If we have a function of multiple variables, say

$$f(a_1, a_2, a_3, \dots),$$

we might care about the change of f with respect to only one variable. By picking a fixed value for all but one of the variables, we can determine this.

Say that we want to find the partial derivative of f with respect to a_2; then, by constructing $g(a_2) = f(c_1, a_2, c_3, \dots)$, we've created a function of *one* variable, which we can differentiate as usual.

Notation 1

We write the partial derivative of a function f at a point $\mathbf{p}$ with respect to a basis element a of $\mathbf{p}$ as $\boxed{f_a(\mathbf{p})}$.

We may also use much more common notation

$$\frac{\partial f}{\partial a},$$

using the partial derivative symbol ∂, a stylized cursive "d".[a]

In the interest of completeness, I'll exhaustedly note that the book also uses, on occasion, the notation $D_a f$.

[a] Introduced by Marquis de Condorcet in 1770, who used it to represent a partial *differential*, i.e. the dy or dx in dy/dx, and then adapted in 1786 by Adrien-Marie Legendre for use as the partial derivative.

We can also calculate higher partial derivatives — similarly to the higher ordinary derivatives. The notation is a fairly clear extension:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}.$$

Theorem 1: Clairaut's Theorem

Suppose f is defined on a neighborhood N about a point $\mathbf{p}$. If f_{xy} and f_{yx} are continuous in N, then $f_{xy}(\mathbf{p}) = f_{yx}(\mathbf{p})$.

2.1 Gradients

Notation 2

This ridiculous textbook denotes the partial derivative of a function $f(x, y) = z$ with respect to x as $f_x(x, y)$.

Definition 1

The **directional derivative** of a function f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h},$$

if the limit exists.

If $f : \mathfrak{R}^2 \mapsto \mathfrak{R}$ is a differentiable function, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ of

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

Or, if $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$, then

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

Noticing that the directional derivative of a function can be written as the dot product of two vectors,

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= f_x(x, y)a + f_y(x, y)b \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle a, b \rangle \\ &= \langle f_x(x, y), f_y(x, y) \rangle \cdot \mathbf{u}, \end{aligned}$$

we call the first vector $\langle f_x(x, y), f_y(x, y) \rangle$ the **gradient** of f and denote it as ∇f.

Definition 2

The gradient of a function f of two variables is defined as

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Therefore, we can rewrite the directional derivative of a function f as

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

It's intuitive, then, that the maximum value of the directional derivative is $|\nabla f(x, y)|$, when $\mathbf{u}$ is parallel to $\nabla f(x, y)$.

2.2 Maximum and minimum values

Definition 3

$f : A^k \mapsto B$ has a **local maximum** at $\mathbf{a}$ if for some neighborhood $N \subset A$ about $\mathbf{a}$, for all $\mathbf{x} \in N$, $f(\mathbf{x}) \leq f(\mathbf{a})$.

Conversely, if $f(\mathbf{x}) \geq f(\mathbf{a})$, then $f(\mathbf{a})$ is a **local *minimum***.

If the statement also holds true for $N = A$, then $\mathbf{a}$ is an **absolute maximum** (or **absolute minimum**).

If f has a local maximum or minimum at $\mathbf{a}$ and the partials of f exist at $\mathbf{a}$, then $\partial f/\partial x(\mathbf{a}) = 0$ and $f_y(a, b) = 0$; geometrically, the tangent plane to a maximum or minimum must be horizontal.

Definition 4

A point $\mathbf{a}$ is called a **critical point** of f if $f_x(\mathbf{a}) = 0$ or $f_x(\mathbf{a})$ doesn't exist for all variables of f.

Definition 5

A **saddle point** of a function is a critical point which is not a local extremum of the function.

If (a, b) is a critical point of f, then let

$$D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - (f_{xy}(a, b))^2.$$

If $D < 0$, then (a, b) is a saddle point of f.

2.3 Lagrange multipliers

Often we want to find the local extrema of a function subject to constraints, i.e. maximizing the volume of an object while keeping its surface area constant. The method of Lagrange multipliers[1] is a strategy for doing this.

To find extrema of $f(\mathbf{p})$ constrained with $g(\mathbf{p}) = k$, we look for extrema of f that are restricted to lie on the level curve $g(\mathbf{p}) = k$; it happens that the largest c such that $f(\mathbf{p}) = c$ intersects with $g(\mathbf{p}) = k$ when the two level curves are tangent with each other, i.e. they have identical normals. In other words, for some scalar λ, $\nabla f(\mathbf{p}) = \lambda \nabla g(\mathbf{p})$.

[1] After Joseph-Louis Lagrange (1736–1813), "an Italian Enlightenment Era mathematician and astronomer [who] made significant contributions to the fields of analysis, number theory, and both classical and celestial mechanics."

More formally, suppose f has an extrema at $\mathbf{p}_0$. Then, let the level surface generated by the constraint $g(\mathbf{p}) = k$ be called S, where $\mathbf{p}_0 \in S$. Then, let C be the set of points given by $\mathbf{r}(t)$ such that $C \subset S$ and $\mathbf{p}_0 \in C$. Further, let t_0 be a point such that $\mathbf{r}(t_0) = \mathbf{p}_0$.

Then, $f \circ \mathbf{r}$ gives the values of f on the curve C. f has an extrema at $\mathbf{p}$, so $f \circ \mathbf{r}$ must also, and $(f \circ \mathbf{r})'(t_0) = 0$. If f is differentiable, we can use the chain rule to write

$$\begin{aligned} 0 &= (f \circ \mathbf{r})'(t_0) \\ &= \nabla f(\mathbf{p}_0) \cdot \mathbf{r}'(t_0). \end{aligned}$$

Therefore, the gradient of f is orthogonal to the tangent of every such curve C. We also know that $\nabla g(\mathbf{p}_0)$ is orthogonal to $\mathbf{r}'(t_0)$, so the gradients of f and g at $\mathbf{p}_0$ must be parallel. Therefore, if $\nabla g(\mathbf{p}_0) \neq 0$, there exists some λ such that

$$\nabla f(\mathbf{p}_0) = \lambda \nabla g(\mathbf{p}_0), \tag{2.1}$$

where the constant λ is called a Lagrange multiplier.

Then, the method of Lagrange multipliers gives us a process to find the maximum and minimum values of a function $f(\mathbf{p})$ subject to the constraint $g(\mathbf{p}) = k$, where $\mathbf{p} \in \mathfrak{R}^n$. To use the method of Lagrange multipliers, we assume that the extreme values exist and that $\nabla g \neq 0$ on the level surface $g(\mathbf{p}) = k$.

1. Find all values of $\mathbf{p}$ and λ such that

$$\begin{aligned} \nabla f(\mathbf{p}) &= \lambda \nabla g(\mathbf{p}) \\ \text{and} \qquad g(\mathbf{p}) &= k. \end{aligned}$$

2. Next, evaluate f at all of the points found in the first step. The largest of these values is the maximum value of f, and the smallest of them is the minimum value.

3 Multiple integrals

Single integrals are good for functions of one variable. To integrate functions of multiple variables, we use multiple integrals. Straightforward enough.

Multiple integrals allow us to calculate things like surface areas and volumes of geometric objects.

In general, for some double integral

$$\underbrace{\int_a^b \overbrace{\int_c^d f(x,y)\,dx}^{\text{We treat } y \text{ as constant while evaluating this.}}\,dy}_{\text{We've eliminated } x \text{ from the equation before evaluating this.}},$$

we do the opposite of partial differentiation and treat all variables other than the one we're integrating for as constant, repeatedly, until we've integrated with respect to all variables; each step in this process is called, predictably, partial integration.

3.1 Double integrals

For an axis-aligned rectangle R on the xy-plane from (x_0, y_0) to (x_1, y_1), the area of a function $f(x,y)$ under R is given by the double integral

$$\iint_Y f(x,y)\,dA = \underbrace{\int_{x_0}^{x_1}\int_{y_0}^{y_1} f(x,y)\,dy\,dx}_{\text{This is the iterated form of the integral.}}$$
$$= \int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)\,dx\,dy,$$

where we use $\iint_R$ to mean "integrating over the area of R" and "dA" to mean "with respect to area."

The right-hand side of the equation above is called the iterated form, or an iterated integral.

We can also iterate over funkier regions if we're willing to play with the limits of integration a bit. The easiest regions to integrate over are the ones that are easily expressible as the region bounded above and below by functions of one variable,

e.g. "the region under the line $y = 2x$ and above the line $y = x^2$" (note that this is bounded on the left at $x = 0$ and on the right at $x = 2$).

The area of that region is expressed by the integral

$$\begin{aligned} A &= \int_0^2 \int_{x^2}^{2x} dy\, dx \\ &= \int_0^2 \Big[x \Big]_{x^2}^{2x} dx \\ &= \int_0^2 \left(2x - x^2 \right) dx \\ &= \left[x^2 - \frac{x^3}{3} \right]_0^2 \\ &= 4 - \frac{8}{3} = \frac{4}{3}. \end{aligned}$$

A more complicated region might be "the region under the paraboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by $y = \sqrt{x}$ and $y = 1 - \cos x$."

We can build larger regions out of pieces, by summing smaller integrals.

3.2 Polar coordinates

Use the conversions

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

for the coordinates and then we have that if R is a "polar rectangle" (arc-shaped region bounded by angles and radii) from $r = a$ to $r = b$ and $\theta = \alpha$ to $\theta = \beta$, we have

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r \cos \theta, r \sin \theta) r\, dr\, d\theta,$$

which makes our lives easier for circly areas and volumes. Don't forget to multiply by r.

For squiggly and varying radii, we can use functions $h_1(\theta)$ and $h_2(\theta)$ instead of constants a and b:

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r\, dr\, d\theta.$$

3.3 Cylindrical coordinates

Just add z.

3.4 Spherical coordinates

I can never remember how these work. If we have a point P, and we drop it down to the xy-plane, the angle between the positive x-axis and the segment from the origin to P is θ.

Next, the angle between the positive z-axis and the segment from the origin to P is ϕ.

Finally, the length of the segment from the origin to P is ρ.

The conversions

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \end{aligned}$$

give us the integral-conversion for the spherical wedge bounded by $a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d$ as

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\theta)[\rho^2\sin\phi]\,d\rho\,d\theta\,d\phi.$$

Very gross!

3.5 Surface area

For $f(x,y)$ with f_x, f_y continuous, the surface area of f within a region D is

$$A = \iint_D \left(\sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\right) dA.$$

4 Vector calculus

A vector field is a mapping $\mathbb{R}^k \mapsto \mathbb{R}^n$; for each point in k-dimensional Euclidean space, we associate an n-dimensional vector. These vectors can represent velocity, distance, or anything else, and come up in all sorts of applied fields.

We'll be mostly concerned with vector fields $\mathbb{R}^2 \mapsto \mathbb{R}^2$ and $\mathbb{R}^3 \mapsto \mathbb{R}^3$.

If we have a plane curve given by the vector equation

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle \qquad a \leq t \leq b,$$

then the line integral of f along $\mathbf{r}(t)$ from a to b is

$$\int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt,$$

i.e. the length of the curve multiplied, at each point, by the value of the vector field f at that point.

A Common formulas for derivatives and integrals

Derivatives

$$\frac{d}{dx}(f+g) = f' + g'$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}(fg) = fg' + f'g$$

$$\frac{d}{dx}\frac{h}{l} = \frac{lh' - hl'}{l^2}$$

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) \quad \text{(Chain rule.)}$$

$$\frac{d}{dx}b^x = b^x \ln b$$

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x)}$$

$$\frac{d}{dx}c = 0$$

$$\frac{d}{dx}cf = cf'$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)} \quad \text{(By the chain rule.)}$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\log_b x = \frac{1}{x \ln b}$$

$$\frac{d}{dx}[\mathbf{u} \cdot \mathbf{v}] = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}'$$

$$\frac{d}{dx}[\mathbf{u} \times \mathbf{v}] = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'$$

Trigenometric

$$\frac{d}{dx} \sin x = \cos x$$
$$\frac{d}{dx} \cos x = -\sin x$$
$$\frac{d}{dx} \tan x = \sec^2 x$$
$$\frac{d}{dx} \cot x = -\csc^2 x$$
$$\frac{d}{dx} \sec x = \sec x \tan x$$
$$\frac{d}{dx} \csc x = -\csc x \cot x$$
$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$
$$\frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2}$$
$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx} \csc^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}$$

Integrals

See also: *Techniques of Integration*.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad \text{when } n \neq -1$$

$$\int x^{-1} \, dx = \ln |x| + C$$

$$\int e^x \, dx = e^x + C$$

$$\frac{d}{dt} \int_{a(t)}^{b(t)} g(s) \, ds = b'(t)g(b(t)) - a'(t)g(a(t)) \quad \text{(Leibniz' rule.)}$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Trigenometric

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{1+x^2}} \, dx = \sin^{-1} x + C$$